A METHOD FOR THE CONSTRUCTION OF LIAPUNOV FUNCTIONS FOR LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS

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Effective methods for the construction of Liapunov functions for systems of linear differential equations with variable coefficients have been developed in the works of Chetaev [1]. The boundaries of the stability region, obtained with the help of these methods, have been studied by Razumikhin [2].

A method for the construction of Liapunov functions for systems of the stated form is proposed below which is related to the above methods.

1. Consider the system of linear differential equations

$$\sum_{k=1}^{n} \varphi_{jk}(D)\, x_k = 0 \qquad (j = 1, \ldots, n) \qquad \left(D = \frac{d}{dt}\right) \tag{1.1}$$

where

$$\varphi_{jk}(D) = b_{jk}^{(0)}(t)\, D^L + b_{jk}^{(1)}(t)\, D^{L-1} + \ldots + b_{jk}^{(L-1)}(t)\, D + b_{jk}^{(L)}(t) \tag{1.2}$$

Introducing the functions

$$l_{jk}^{(s)}(t) = b_{jk}^{(s)}(t) - a_{jk}^{(s)} \qquad (a_{jk}^{(s)} = \text{const}) \tag{1.3}$$

the operators $\phi_{jk}(D)$ may be represented in the form

$$\varphi_{jk}(D) = f_{jk}(D) + L_{jk}(D) \tag{1.4}$$

where

$$f_{jk}(D) = a_{jk}^{(0)} D^L + a_{jk}^{(1)} D^{L-1} + \ldots + a_{jk}^{(L-1)} D + a_{jk}^{(L)} \tag{1.5}$$

$$L_{jk}(D) = l_{jk}^{(0)}(t)\, D^L + l_{jk}^{(1)}(t)\, D^{L-1} + \ldots + l_{jk}^{(L-1)}(t)\, D + l_{jk}^{(L)}(t) \tag{1.6}$$

The system of differential equations (1.1) may now be written

$$\sum_{k=1}^{n} f_{jk}(D)\, x_k = -\sum_{k=1}^{n} L_{jk}(D)\, x_k \qquad (j = 1, \ldots, n) \tag{1.7}$$

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Together with the system of equations (1.7), consider the system of linear, non-homogeneous equations with constant coefficients

$$\sum_{k=1}^{n} f_{jk}(D)\, x_k = y_j(t) \qquad (j = 1, \ldots, n) \tag{1.8}$$

Denote by $\Delta(D)$ the determinant of the operator matrix $f(D)$

$$\Delta(D) = \begin{vmatrix} f_{11}(D) & \ldots & f_{n}\ (D) \\ \ldots & \ldots & \ldots \\ f_{n1}(D) & \ldots & f_{nn}(D) \end{vmatrix} \tag{1.9}$$

The roots of the characteristic equation

$$\Delta(D) = 0 \tag{1.10}$$

will be denoted as follows: the real roots by $\kappa_g (g = 1, \ldots, N')$ the conjugate complex roots by $\epsilon_h \pm i\omega_h (h = N' + 1, \ldots, N' + N'')$ the total number of roots by $N = N' + 2N''$. For the sake of simplicity, we will assume that all the roots of the characteristic equation are simple. (One could also have admitted the presence of multiple roots, but with linear elementary divisors.)

The system of differential equations (1.8) may be transformed by transition from the original coordinates x_j to the normal coordinates ξ_g, ξ_h, η_h $(g = 1, \ldots, N', h = N' + 1, \ldots, N' + N'')$. The formulas, relating the original and the normal coordinates will be as follows [3]:

$$\overset{\vartheta}{x}_j = \sum_{g=1}^{N'} X_{j\vartheta}{}^{(g)}\xi_g + \sum_{h=N'+1}^{N'+N''} (X_{j\vartheta}{}^{(h)}\xi_h + Y_{j\vartheta}{}^{(h)}\eta_h) \qquad \begin{pmatrix} j = 1,2, \ldots, n \\ \vartheta = 0,1,\ldots, m_j - 1 \end{pmatrix} \tag{1.11}$$

$$X_{j\vartheta}{}^{(g)} = X_j{}^{(g)}\kappa_g{}^{\vartheta}, \qquad X_{j\vartheta}{}^{(h)} = N_{j\vartheta}{}^{(h)}\cos(\gamma_j{}^{(h)} + \vartheta\zeta_h)$$

$$Y_{j\vartheta}{}^{(h)} = N_{j\vartheta}{}^{(h)}\sin(\gamma_j{}^{(h)} + \vartheta\zeta_h) \tag{1.12}$$

where $N_{j\theta}{}^{(h)} = N_j{}^{(h)} c_h{}^{\theta}$; the quantities c_h and ζ_h are determined by the

$$\varepsilon_h + i\omega_h = c_h e^{i\zeta_h} \tag{1.13}$$

The quantity $X_j{}^{(g)}$ is the j'th element of the non-zero column X_g of the adjoined matrix $F(\kappa_g)$, constructed for the real root κ_g. The quantity $N_j{}^{(h)} e^{i\gamma_j{}^{(h)}} = X_j{}^{(h)} + iY_j{}^{(h)}$ is the j'th element of the non-zero column X_h of the adjoined matrix $F(\epsilon_h + i\omega_h)$, constructed for the complex root $\epsilon_h + i\omega_h$. The quantity m_j is the order of the highest derivative of x_j occurring in the system of differential equations (1.8). As has been shown in the cited paper [3] by Bulgakov, the normal coordinates ξ_g, ξ_h, η_h satisfy the following system of differential equations:

$$\frac{d\xi_g}{dt} = \varkappa_g\xi_g + \left[\frac{D-\varkappa_g}{\Delta(D)}\right]_{D=\varkappa_g} \sum_{k=1}^{n} B_k^{(g)} y_k(t) \qquad (g=1,\ldots,N')$$

$$\frac{d\xi_h}{dt} = \varepsilon_h\xi_h - \omega_h\eta_h + 2\mathrm{Re}\left[\frac{D-\varepsilon_h-i\omega_h}{\Delta(D)}\right]_{D=\varepsilon_h+i\omega_h} \sum_{k=1}^{n} B_k^{(h)} y_k(t) \tag{1.14}$$

$$\frac{d\eta_h}{dt} = \varepsilon_h\eta_h - \omega_h\xi_h - 2\mathrm{Im}\left[\frac{D-\varepsilon_h-i\omega_h}{\Delta(D)}\right]_{D=\varepsilon_h+i\omega_h} \sum_{k=1}^{n} B_k^{(h)} y_k(t) \quad (h=N'+1,\ldots,N'+N'')$$

Here $B_k^{(g)}$ are the elements of the row matrix B_g and, analogously, $B_k^{(h)}$ the elements of the row matrix B_h. These row matrices are introduced in order to satisfy the relations

$$F(\varkappa_g) = X_g B_g, \qquad F(\varepsilon_h + i\omega_h) = X_h B_h \tag{1.15}$$

In an analogous manner, one may also construct equations in normal coordinates for the system (1.7). For this purpose, we transform the right-hand sides of the equations (1.7), replacing in them $\underset{k}{\overset{\theta}{x}}$ by the expressions (1.11), which in this way become

$$-\sum_{k=1}^{n} L_{jk}(D)\, x_k = \sum_{g=1}^{N'} \mu_{jg}(t)\,\xi_g + \sum_{h=N'+1}^{N'+N''} [\mu_{jh}(t)\,\xi_h + \nu_{jh}(t)\,\eta_h] \tag{1.16}$$

The functions (1.16) depend linearly on the normal coordinates ξ_g, ξ_h, η_h; for the sake of brevity, we will denote them by

$$\sum_{g=1}^{N'} \mu_{jg}(t)\,\xi_g + \sum_{h=N'+1}^{N'+N''} [\mu_{jh}(t)\,\xi_h + \nu_{jh}(t)\,\eta_h] \equiv \Lambda_j(\xi_g,\ \xi_h,\ \eta_h,\ t) \tag{1.17}$$

Substituting in (1.14) for $y_j(t)$ the functions (1.17), we obtain the system of equations in normal coordinates, equivalent to the original system of differential equations (1.1)

$$\frac{d\xi_g}{dt} = \varkappa_g\xi_g + \left[\frac{D-\varkappa_g}{\Delta(D)}\right]_{D=\varkappa_g} \sum_{k=1}^{n} B_k^{(g)} \Lambda_k(\xi_g,\ \xi_h,\ \eta_h,\ t) \qquad (g=1,\ldots,N')$$

$$\frac{d\xi_h}{dt} = \varepsilon_h\xi_h + \omega_h\eta_h + 2\mathrm{Re}\left[\frac{D-\varepsilon_h-i\omega_h}{\Delta(D)}\right]_{D=\varepsilon_h+i\omega_h} \sum_{k=1}^{n} B_k^{(h)} \Lambda_k(\xi_g, \xi_h, \eta_h, t) \tag{1.18}$$

$$\frac{d\eta_h}{dt} = \varepsilon_h\eta_h - \omega_h\xi_h - 2\mathrm{Im}\left[\frac{D-\varepsilon_h-i\omega_h}{\Delta(D)}\right]_{D=\varepsilon_h+i\omega_h} \sum_{k=1}^{n} B_k^{(h)} \Lambda_k(\xi_g,\ \xi_h,\ \eta_h,\ t)$$

$$(h = N'+1,\ldots,N'+N'')$$

Like the original system (1.1), the system (1.18) is a system of linear differential equations with variable coefficients. However, for the system of equations (1.18), one may give a simple method for the construction of Liapunov functions which leads to sufficient conditions for the stability of the trivial solution of this system.

We seek the Liapunov function in the form

$$V = -\frac{1}{2}\left[\sum_{g=1}^{N'} \xi_g^2 + \sum_{h=N'+1}^{N'+N''} (\xi_h^2 + \eta_h^2)\right] \tag{1.19}$$

The function V is positive definite. Its derivative with respect to time

$$\dot{V} = -\sum_{g=1}^{N'} \xi_g \frac{d\xi_g}{dt} - \sum_{h=N'+1}^{N'+N''} \left(\xi_h \frac{d\xi_h}{dt} + \eta_h \frac{d\eta_h}{dt}\right) \tag{1.20}$$

after replacing $d\xi_g/dt$, $d\xi_h/dt$, $d\eta_h/dt$ by their values (1.18), will itself be a quadratic form in the variables ξ_g, ξ_h, η_h :

$$\dot{V} = \sum_{g=1}^{N'} [-\varkappa_g + \Psi_{gg}(t)]\xi_g^2 + \sum_{h=N'+1}^{N'+N''} \{[-\varepsilon_h + \Psi_{hh}^*(t)]\xi_h^2 + \tag{1.21}$$

$$+ [-\varepsilon_h + \Psi_{hh}^{**}(t)]\eta_h^2\} + 2c_{12}\xi_1\xi_2 + 2c_{13}\xi_1\xi_3 + \ldots + 2c_{1,N-1}\xi_1\xi_{N'+N''} +$$
$$+ 2c_{1N}\xi_1\eta_{N'+N''} + 2c_{23}\xi_2\xi_3 + \ldots + 2c_{2,N-1}\xi_2\xi_{N'+N''} + 2c_{2N}\xi_2\eta_{N'+N''} +$$
$$+ \ldots + 2c_{N-1,N}\xi_{N'+N''}\eta_{N'+N''}$$

The coefficients $c_{ij} (i \neq j)$ of the quadratic form (1.21) are combinations of the original variable coefficients $l_{jk}^{(s)}(t)$. If all the original coefficients $l_{jk}^{(s)}(t) \equiv 0$, the derivative $\dot{V}$ takes the form

$$\dot{V} = -\sum_{g=1}^{N'} \varkappa_g \xi_g^2 - \sum_{h=N'+1}^{N'+N''} \varepsilon_h (\xi_h^2 + \eta_h^2) \tag{1.22}$$

In the case when all the roots of the characteristic equation (1.10) lie in the left-half-plane, i.e. when all $\kappa_g < 0$, $\epsilon_h < 0$, the derivative $\dot{V}$ is itself a positive definite function (i.e., its sign is opposite to that of the function V), as must be the case for an asymptotically stable system.

The quadratic form (1.21) has the discriminant

$$\Delta = \begin{vmatrix} c_{11} & \cdots & c_{1N} \\ \cdot & \cdots & \cdot \\ c_{N1} & \cdots & c_{NN} \end{vmatrix} \tag{1.23}$$

where

$$c_{11} = -\varkappa_1 + \Psi_{11}(t), \ldots, \quad c_{NN} = -\varepsilon_{N'+N''} + \Psi_{N'+N'',N'+N''}^{**}(t) \tag{1.24}$$

The conditions for $\dot{V}$ to have a definite (positive) sign are that all principal, diagonal minors of the discriminant (1.23) must be positive at any instant of the time t. These conditions are also sufficient conditions for the stability of the trivial solution of the system (1.1) of differential equations with variable coefficients.

Since the above stability conditions are sufficient, and not necessary, we note that by varying the **form** of V one may sometimes extend the stability region, obtained from the conditions for the function V to have only one sign, in the space of the parameters of the system. In order to vary the Liapunov function, one may take it in the form

$$V = -\frac{1}{2}\left[\sum_{g=1}^{N'} p_g \xi_g^2 + \sum_{h=N'+1}^{N'+N''} (p_h \xi_h^2 + q_h \eta_h^2)\right] \qquad (1.25)$$

where the coefficients p_g, p_h, q_h must be strictly positive. The choice of the values of the coefficients p_g, p_h, q_h may be subjected to a definite requirement, for example, that any coefficients c_{rs} in the quadratic form (1.21) must become zero, etc.

The choice of the coefficients $a_{jk}^{(s)}$ in the expressions (1.3) must also be subjected to the requirement of the maximum extension of the stability region; the values of the coefficients $a_{jk}^{(s)}$ are conveniently selected in such a manner that the region of stability in the space of the interesting parameters, obtained by help of the Liapunov function, will be as large as possible.

2. As an example, consider the system of differential equations

$$ax_1 + \dot{x}_2 = 0, \qquad \dot{x}_1 - \frac{b}{a}x_2 - \frac{bk}{a}x_3 + \frac{bk}{a}x_4 = 0$$

$$-\mu(t)x_1 + cx_2 + \dot{x}_3 + cx_3 = 0, \qquad \dot{x}_4 + cx_4 = 0 \qquad (2.1)$$

Assuming the function $\mu(t)$ to be bounded and denoting by fa its largest absolute value $|\mu(t)| \leqslant fa$, the system (2.1) may be rewritten

$$ax_1 + \dot{x}_2 = 0, \qquad \dot{x}_1 - \frac{b}{a}x_2 - \frac{bk}{a}x_3 + \frac{bk}{a}x_4 = 0 \qquad (2.2)$$

$$-fax_1 + cx_2 + \dot{x}_3 + cx_3 = s(t)x_1, \qquad \dot{x}_4 + cx_4 = 0$$

In this system, $-2fa \leqslant s(t) = \mu(t) - fa \leqslant 0$. For $s(t) \equiv 0$, the system (2.2) becomes a system of linear equations with constant coefficients with the following characteristic equation:

$$(D + c)[D^3 + cD^2 + b(1 - fk)D + (1 - k)bc] = 0 \qquad (2.3)$$

The roots of the algebraic equation (2.3) will lie in the left half-plane of the complex variable D, provided $f < 1$. In this context, it has been assumed that the coefficients a, b, c, k of the system of equations (2.1) are positive and, in addition, that $k < 1$.

We confine ourselves to the case when the coefficients in (2.3) are such that this equation has two real and one pair of conjugate complex roots. Denote these roots by κ_1, κ_2, $\epsilon \pm i\omega$, where $\kappa_1 = -c$. The quantities κ_2, ϵ and ω will then satisfy the relation

$$(D-\varkappa_2)(D-\varepsilon-i\omega)(D-\varepsilon+i\omega)=D^3+cD^2+b(1-f/k)D+(1-k)bc \quad (2.4)$$

In order to transform the system (2.1) to the new variables ξ_1, ξ_2, ξ_3, η_3 which represent the normal coordinates for this system in the case when $s(t)\equiv 0$, one must, in accordance with (1.11), introduce these variables by means of the relations

$$x_1=\frac{\varkappa_2(\varkappa_2+c)}{(f\varkappa_2+c)a}\xi_2+m_1\xi_3+m_2\eta_3, \qquad x_3=\xi_1+\xi_2+\xi_3 \quad (2.5)$$
$$x_2=-\frac{\varkappa_2+c}{f\varkappa_2+c}\xi_2+n_1\xi_3+n_2\eta_3, \qquad x_4=\xi_1$$

where

$$m_1=\frac{1}{al}[f\varepsilon^3+c(1+f)\varepsilon^2+(c^2+f\omega^2)\varepsilon-c\omega^2(1-f)]$$
$$m_2=\frac{\omega}{al}(f\varepsilon^2+2c\varepsilon+c^2+f\omega^2)$$
$$n_1=-\frac{1}{l}[f\varepsilon^2+c(1+f)\varepsilon+c^2+f\omega^2] \quad (2.6)$$
$$n_2=-\frac{\omega c}{l}(1-f), \qquad l=f^2\varepsilon^2+2cf\varepsilon+c^2+f^2\omega^2$$

The differential equations, satisfied by the new variables ξ_1, ξ_2, ξ_3, η_3 will by (1.18) have the form

$$\frac{d\xi_1}{dt}=\varkappa_1\xi_1$$
$$\frac{d\xi_2}{dt}=[\varkappa_2+a_2s(t)]\xi_2+a_3s(t)\xi_3+a_4s(t)\eta_3$$
$$\frac{d\xi_3}{dt}=b_2s(t)\xi_2+[\varepsilon+b_3s(t)]\xi_3+[\omega+b_4s(t)]\eta_3 \quad (2.7)$$
$$\frac{d\eta_3}{dt}=c_2s(t)\xi_2+[-\omega+c_3s(t)]\xi_3+[\varepsilon+c_4s(t)]\eta_3$$

where

$$a_2=A_2\frac{\varkappa_2(\varkappa_2+c)}{(f\varkappa_2+c)a}, \qquad a_3=A_2m_1, \qquad a_4=A_2m_2$$
$$b_2=A_3\frac{\varkappa_2(\varkappa_2+c)}{(f\varkappa_2+c)a}, \qquad b_3=A_3m_1, \qquad b_4=A_3m_2 \quad (2.8)$$
$$c_2=A_4\frac{\varkappa_2(\varkappa_2+c)}{(f\varkappa_2+c)a}, \qquad c_3=A_4m_1, \qquad c_4=A_4m_2$$

$$A_2=T_2(f\varkappa_2+c)bk$$
$$A_3=[(f\varepsilon+c)T_3+f\omega T_4]bk \quad (2.9)$$
$$A_4=[(f\varepsilon+c)T_4-f\omega T_3]bk$$

$$T_2 = \frac{1}{(\varkappa_2 + c)\,[(\varkappa_2 - \varepsilon)^2 + \omega^2]}$$

$$T_3 = \frac{-(2\varepsilon - \varkappa_2 + c)}{[(\varepsilon + c)(\varepsilon - \varkappa_2) - \omega^2]^2 + [(2\varepsilon - \varkappa_2 + c)\,\omega]^2} \qquad (2.10)$$

$$T_4 = \frac{(\varepsilon + c)(\varepsilon - \varkappa_2) - \omega^2}{\omega\,\{[(\varepsilon + c)(\varepsilon - \varkappa_2) - \omega^2]^2 + [(2\varepsilon - \varkappa_2 + c)\,\omega]^2\}}$$

Select the Liapunov function in the form

$$V = -\tfrac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 + \eta_3^2) \qquad (2.11)$$

Its derivative with respect to time

$$V = -\left(\xi_1 \frac{d\xi_1}{dt} + \xi_3 \frac{d\xi_2}{dt} + \xi_2 \frac{d\xi_3}{dt} + \eta_3 \frac{d\eta_3}{dt}\right)$$

after substitution of the expressions for the derivatives (2.7), takes the form

$$V = -\varkappa_1\xi_1^2 - [\varkappa_2 + a_2 s(t)]\,\xi_2^2 - [\varepsilon + b_3 s(t)]\,\xi_3^2 - [\varepsilon + c_4 s(t)]\,\eta_3^2 -$$
$$-(a_3 + b_2)\,s(t)\,\xi_2\xi_3 - (a_4 + c_2)\,s(t)\,\xi_2\eta_3 - (b_4 + c_3)\,s(t)\,\xi_3\eta_3 \qquad (2.12)$$

The discriminant of the quadratic form (2.12) is

$$Д = \begin{vmatrix} -\varkappa_1 & 0 & 0 & 0 \\ 0 & -\varkappa_2 - a_2 s(t) & \dfrac{-(a_3 + b_2)}{2} s(t) & \dfrac{-(a_4 + c_2)}{2} s(t) \\ 0 & \dfrac{-(a_3 + b_2)}{2} s(t) & -\varepsilon - b_3\ s(t) & \dfrac{-(b_4 + c_3)}{2} s(t) \\ 0 & \dfrac{-(a_4 + c_2)}{2} s(t) & \dfrac{-(b_4 + c_3)}{2} s(t) & -\varepsilon - c_4\ s(t) \end{vmatrix} \qquad (2.13)$$

The conditions for asymptotic stability of the trivial solution of the system of equations (2.1) are that at any instant of time t all the principal diagonal minors of the discriminant (2.13) must be positive.

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